Lecture 21

Tuesday, 21 April 2026 1:29 AM

Generalised Assignment Problem (GAP)

m jobs, to be processed on n machines.
job j requires p_{ij} processing time on machine i
each job to be assigned to a single machine.

Let $A = (A_1, \dots, A_n)$ be an assignment of jobs to machines, A_i is set of jobs assigned to machine i.
Then machine i takes time $\sum_{j \in A_i} p_{ij}$ to complete all jobs assigned to it.
The <u>makespan</u> of the assignment A is $\max_i \sum_{j \in A_i} p_{ij}$

Problem (Minimum Makespan Scheduling): Given a makespan T, return an assignment w/ makespan $\leq T$ if it exists.

Generalized assignment problem: Assigning job j to machine i also incurs cost c_{ij}.
Total <u>cost</u> of assignment A is $\sum_i \sum_{j \in A_i} c_{ij}$

Problem: Given bounds C & T on total cost & makespan, return an assignment of total cost $\leq C$ & makespan $\leq T$, if it exists.

Example: (p_{ij}, c_{ij})

m/c				
1	8, 10	6, 10	12, 4	8, 20
2	9, 1	9, 1	15, 20	4, 10

what is min total cost? min makespan?
fractional assignments?

(Shmoys, Tardos '93)

Theorem: There is a poly-time algo that given C, T, if there exists such an assignment, then it returns an assignment w/ total cost $\leq C$ & cost $\leq 2T$.

(Lenstra, Shmoys, Tardos '90)

Lower bounds: Approximating the optimal makespan within a $3/2$ factor is NP-hard
(reduction from 3-DIM matching, try yourself)

Algorithm:

$$\min \sum_{i,j} c_{ij} x_{ij}$$
$$\text{s.t. } \forall i, \quad \sum_j p_{ij} x_{ij} \leq T$$
$$\forall j, \quad \sum_i x_{ij} \geq 1$$
$$\forall i,j, \quad x_{ij} \geq 0$$

Unfortunately, this LP has a large integrality gap of m.

eg. m machines, 1 job, showing (p_{ij}, c_{ij}):

m/c	
1	$(m, 1)$
2	⋮
⋮	⋮
m	$(m, 1)$

, then optimal integral soln. has makespan m
optimal fractional has makespan 1.

We use the fact that if there is a schedule of makespan T, job j cannot be assigned to machine i if $p_{ij} > T$.

LP(T):

$$\min \sum_{i,j} c_{ij} x_{ij}$$
$$\text{s.t. } \forall i \quad \sum_j p_{ij} x_{ij} \leq T$$
$$\forall j \quad \sum_i x_{ij} = 1$$
$$\forall i,j \quad x_{ij} \geq 0$$
$$\forall i,j \quad x_{ij} = 0 \quad \text{if } p_{ij} > T$$

Let x be an optimal soln to LP(T).
$\forall i$, $k_i := \lceil \sum_j x_{ij} \rceil$ be the total fraction (rounded up) of jobs assigned to i.

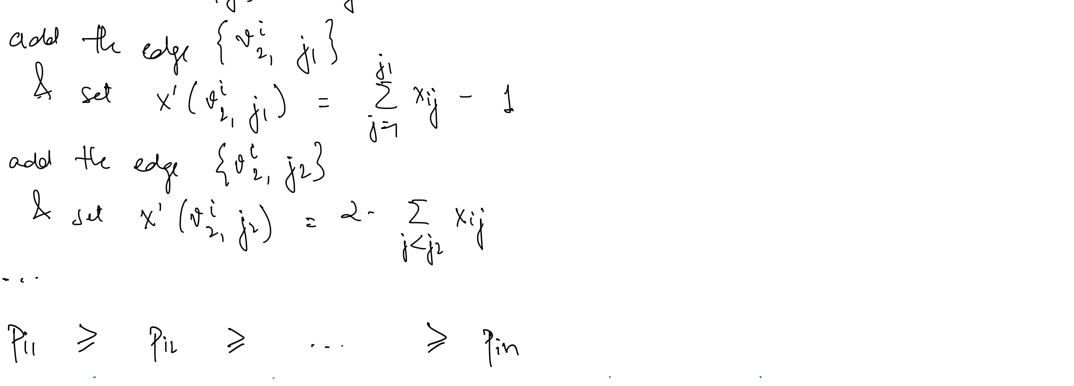
Construct a bipartite graph $B = (\hat{M} \cup J, E)$, and a fractional matching x' on B as follows:
- J is set of jobs
- $\hat{M}$ consists of $\sum_i p_i$ nodes.
  $\hat{M} = \{ v_s^i : i \in M, s \in [k_i] \}$

For the edges:
Fix machine i. Order jobs by nonincreasing processing time.
(so assume that $p_{i1} \geq p_{i2} \geq \dots \geq p_{in}$)
Find first job j_1 so that $\sum_{j=1}^{j_1} x_{ij} \geq 1$.

For each $j \in \{1, \dots, j_1 - 1\}$ so that $x_{ij} > 0$:
  add the edge $\{v_1^i, j\}$
    & set $x'(v_1^i, j) = x_{ij}$
also add the edge $\{v_1^i, j_1\}$
    & set $x'(v_1^i, j_1) = 1 - \sum_{j < j_1} x_{ij}$

This gives the edges incident on v_1^i. For v_2^i:
Find first job j_2 so that $\sum_{j=1}^{j_2} x_{ij} \geq 2$.
For each $j \in \{j_1 + 1, \dots, j_2 - 1\}$ so that $x_{ij} > 0$,
add the edge $\{v_2^i, j\}$
  & set $x'(v_2^i, j) = x_{ij}$
add the edge $\{v_2^i, j_1\}$
  & set $x'(v_2^i, j_1) = \sum_{j=1}^{j_1} x_{ij} - 1$
add the edge $\{v_2^i, j_2\}$
  & set $x'(v_2^i, j_2) = 2 - \sum_{j < j_2} x_{ij}$
...

$p_{i1} \geq p_{i2} \geq \dots \geq p_{in}$



$p_{1,max}^i = p_{i1}$, $p_{1,min}^i = p_{i2}$, $p_{2,max}^i = p_{i2}$, $p_{2,min}^i = p_{i4}$
Thus, $p_{s,max}^i \leq p_{s-1,min}^i$

Note that:
$$\forall i,j, \quad x_{ij} = \sum_{s \in [k_i]} x'(v_s^i, j)$$

Thus $\forall j$, $1 = \sum_i x_{ij} = \sum_i \sum_{s \in [k_i]} x'(v_s^i, j)$ (each vtx. j is fully matched by x')

& $\forall i, s \in [k_i]$, $\sum_j x'(v_s^i, j) \leq 1$ (& $= 1$ for $s < k_i$)
(each vtx. v_s^i for $s < k_i$ is fully matched in x', & $v_{k_i}^i$ is partially matched).

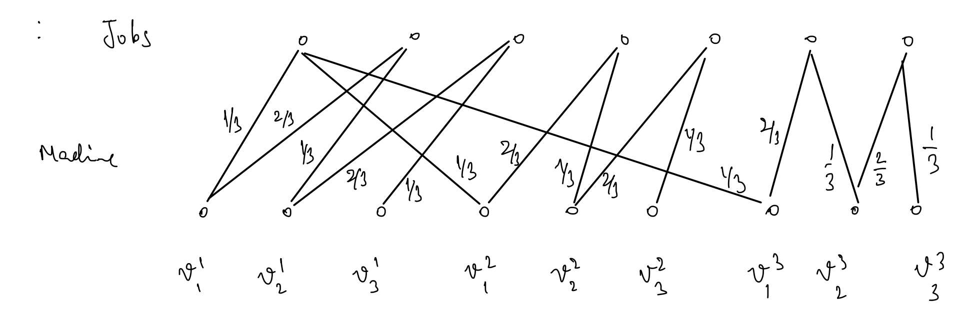
Example:

$p_{ij} = \begin{matrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$

$$X = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 1/3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$k_i = \lceil 7/3 \rceil = 3 \quad \forall i$

B, x' : Jobs



Claim: $\sum_{i,j} c_{ij} x_{ij} = \sum_{i,j} c_{ij} \sum_{s \in [k_i]} x'(v_s^i, j) = \sum_{i,j} \sum_{s \in [k_i]} c_{ij} x'(v_s^i, j)$

Thus, x' is a fractional job-perfect matching in B of cost C.

<u>Algo:</u> Construct the b.p. graph $B = (J \cup \hat{M}, E)$
Find a min-cost job-perfect matching in B
Return the assignment given by the matching.

Let $\bar{x}$ be a min cost job-perfect matching in B.

Claim: The cost of $\bar{x}$ is at most C.

Claim: The makespan of the assignment given by $\bar{x}$ is at most $2T$

Proof: Fix a machine i. In B, i has at most $k_i = \lceil \sum_j x_{ij} \rceil$ nodes, so there are at most k_i jobs assigned to i by $\bar{x}$.
Each node v_s^i, $s = 1 \dots k_i$, can be matched to a job of processing time at most $p_{s,max}^i$. Thus:

$$\sum_j p_{ij} \bar{x}_{ij} = \sum_{s=1}^{k_i} p_{s,max}^i \leq T + \sum_{s=2}^{k_i} p_{s,max}^i$$
$$\leq T + \sum_{s=2}^{k_i - 1} p_{s,min}^i = T + \sum_{s=2}^{k_i - 1} p_{s,min}^i \sum_j x'(v_s^i, j)$$
$$\leq T + \sum_{s=2}^{k_i - 1} \sum_j p_{ij} x'(v_s^i, j) = T + \sum_j p_{ij} \sum_{s=2}^{k_i - 1} x'(v_s^i, j)$$
$$\leq T + \sum_j p_{ij} x_{ij} \leq T$$

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